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TOPOLOGICAL ENTROPY AT AN Ω-EXPLOSION

BY

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ABSTRACT. In this paper an example is given of a C^2 map g from the circle onto itself, which permits an Ω-explosion. It is shown that topological entropy (considered as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at g.

1. **Introduction.** Let g denote any C^2 mapping of the circle onto itself which satisfies the following properties (see Figure 1):

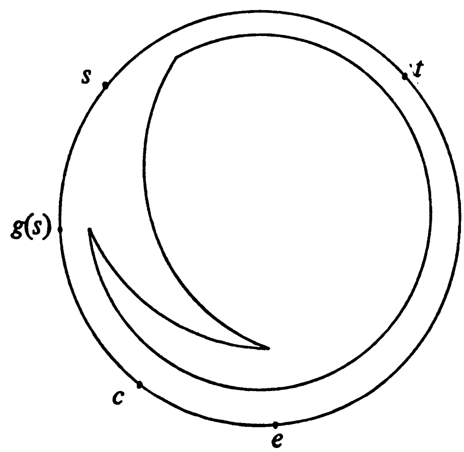

FIGURE 1

(1) g has an expanding fixed point e and a contracting fixed point c, and these are the only fixed points of g.

(2) g preserves orientation at e and c.

(3) g has nondegenerate singularities t and s, and these are the only singularities of g.

(4) The points e, t, s, $g(s)$, and c are distinct and in order on the circle in the counterclockwise direction.

(5) g is one-to-one on each of the intervals (e, t), (t, s), and (s, e). Here we

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use the notation (a, b) to denote the open arc from a counterclockwise to b.

(6) $g(t) = e$.

These properties imply that $\Omega(g) = \{e, c\}$, where $\Omega(g)$ denotes the nonwandering set (see [2], [5], or [7] for definition).

It is easy to see that g permits an Ω-explosion. By this we mean that for any neighborhood N of g in $C^2(S^1, S^1)$, there is a map $f \in N$ with $\Omega(f)$ infinite. See Proposition 9 in §4 for a proof.

Let ent denote topological entropy (see §2 for the definition). Our main result is the following:

THEOREM A. *The map* ent: $C^2(S^1, S^1) \to R$ *is continuous at* g.

Theorem A implies that for any bifurcation through g, at the map g there is no sudden jump in the amount of action.

To prove Theorem A we first, in §3, obtain an upper bound for entropy (Theorem 6). Then, in §4, we construct a sequence (f_n) of maps with $\mathrm{ent}(f_n) \to 0$ as $n \to \infty$. Finally, in §5, we prove Theorem A by using Theorem 6 to show that for arbitrarily large n if f is close enough to g in $C^2(S^1, S^1)$, then $\mathrm{ent}(f) < \mathrm{ent}(f_n)$.

2. **Preliminary definitions and results.** We first review the definition of topological entropy given in [1]. Let X be a compact space and $f: X \to X$ a continuous map. For any two open covers $\mathcal{A}$ and $\mathcal{B}$ of X, let $\mathcal{A} \vee \mathcal{B}$ denote $\{A \cap B: A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$, and let $f^{-1}(\mathcal{A})$ denote $\{f^{-1}(A): A \in \mathcal{A}\}$. Let $M_n(f, \mathcal{A})$ denote the minimum cardinality of a subcover of X of

$$(*) \qquad \mathcal{A} \vee f^{-1}(\mathcal{A}) \vee \cdots \vee f^{(-n+1)}(\mathcal{A}).$$

We set

$$\mathrm{ent}(f, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n}\left(\ln(M_n(f, \mathcal{A}))\right),$$

where ln denotes the natural logarithm. It is easy to see that this limit exists and is finite (see [1]). Finally, we define the topological entropy of f by

$$\mathrm{ent}(f) = \sup(\mathrm{ent}(f, \mathcal{A}))$$

where the supremum is taken over all open covers $\mathcal{A}$ of X. If X is a metric space it suffices to consider any sequence of open covers whose diameter approaches zero (see [1]). By the diameter of an open cover $\mathcal{A}$ we mean the supremum of the diameters of the open sets in $\mathcal{A}$.

We now state some basic facts about topological entropy which will be used later. In each of the four propositions, f is a continuous map of a compact space X into itself. Proposition 2 follows immediately from the definition, and Proposition 4 follows from Proposition 3.

PROPOSITION 1 (SEE [5]). *If X is a metric space,* $\mathrm{ent}(f) = \mathrm{ent}(f|\Omega(f))$.

PROPOSITION 2. *If X is finite,* $\mathrm{ent}(f) = 0$.

PROPOSITION 3 (SEE [1]). *If X_1 and X_2 are closed subsets of X, with $X_1 \cup X_2 = X$ and $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then*

$$\mathrm{ent}(f) = \max\{\mathrm{ent}(f|X_1), \mathrm{ent}(f|X_2)\}.$$

PROPOSITION 4. *If K is a closed subset of X, with $f(K) \subset K$, then* $\mathrm{ent}(f|K) \leqslant \mathrm{ent}(f)$.

We will assume the reader is familiar with the following terminology (see [2] or [7]); nonwandering set, expanding fixed point, contracting fixed point, and stable manifold of a contracting fixed point (denoted $W^s(c)$). For any point $x \in W^s(c)$, we will use the notation $\mathrm{slsm}(x)$ to denote the component of $W^s(c)$ which contains x.

3. **An upper bound for entropy.** The proof of Theorem 6 (which modifies a theorem of [2]) uses the following lemma from [2] (see [2, §3, Lemma 5]).

LEMMA 5. *Let $f \in C^0(S^1, S^1)$. Let $K_1, \ldots, K_n$ be proper closed intervals of S^1, such that for each $i = 1, \ldots, n-1$, $f|K_i$ is a homeomorphism and $f(K_i) \subset K_{i+1}$. Let $\mathcal{A}$ be a covering of $K_1 \cup \cdots \cup K_n$ by finitely many open intervals, such that $\forall A \in \mathcal{A}$ and $i = 1, \ldots, n$, $A \cap K_i$ is an interval (or empty). Let* $\mathrm{card}(\mathcal{A}) = k$ (card *denotes cardinality). Then there is a subset $\mathcal{B}$ of* (*) *which covers K_1, with* $\mathrm{card}(\mathcal{B}) \leqslant n \cdot k$.

In Theorems 6 and A we will use the following definition. Let $\mathcal{C}$ be a finite collection of closed intervals on S^1, $\mathcal{C} = \{I(1), \ldots, I(p)\}$, and let $f \in C^0(S^1, S^1)$. We denote by $K_n(f, \mathcal{C})$ the number of distinct nonempty sets of the form

$$I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{(-n+1)}(I(j_n)),$$

where $j_i \in \{1, \ldots, p\}$ for $i = 1, \ldots, n$.

THEOREM 6. *Let $f \in C^1(S^1, S^1)$, and let $\mathcal{C} = \{I(1), \ldots, I(p)\}$ be a finite collection of proper closed intervals on S^1. Let $W = S^1$ or $W = S^1 - (O_1 \cup O_2 \cup \cdots \cup O_m)$ where O_i is a component of the stable manifold of a contracting periodic point c_i for $i = 1, \ldots, m$. Suppose the following conditions hold.*

(1) $I(1) \cup \cdots \cup I(p) = W$.

(2) *For $j = 1, \ldots, p$, f maps $I(j)$ homeomorphically onto its image.*

(3) *For any $i = 1, \ldots, p$ and $j = 1, \ldots, p$, $f(I(i)) \cap I(j)$ is an interval.*

Then

$$\mathrm{ent}(f) \leqslant \lim_{n\to\infty} \frac{1}{n}\left(\ln(K_n(f, \mathcal{C}))\right).$$

PROOF. Let δ denote the minimum length of the intervals $S^1 - I(j)$ where $j = 1, \ldots, p$. Let $\mathcal{A}$ be any finite cover of $I(1) \cup \cdots \cup I(p)$ by open intervals with the diameter of $\mathcal{Q}$ less than δ. Let $k = \operatorname{card}(\mathcal{Q})$.

Let

$$I(j_1, j_2, \ldots, j_n) = I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{(-n+1)}(I(j_n))$$

where $j_i \in \{1, 2, \ldots, p\}$ $\forall i = 1, \ldots, n$. Then each nonempty $I(j_1, j_2, \ldots, j_n)$ is a closed interval and f maps $I(j_1, j_2, \ldots, j_n)$ homeomorphically into $I(j_2, \ldots, j_n)$. For any fixed $I(j_1, j_2, \ldots, j_n)$, by Lemma 5 (with $K_i = I(j_1, \ldots, j_n)$), there is a subset of (*) which covers $I(j_1, j_2, \ldots, j_n)$ of cardinality at most $n \cdot k$.

Let $X = S^1 - (\bigcup_{i=1}^m W^s(c_i))$. Then X is a compact set with $f(X) \subset X$ and $f^{-1}(X) \subset X$.

Let $\mathcal{Q}(X)$ be the open cover of X defined by $\mathcal{Q}(X) = \{A \cap X: A \in \mathcal{Q}\}$. Then for any $I(j_1, j_2, \ldots, j_n)$, the minimal number of open sets of

$$\mathcal{Q}(X) \vee f^{-1}(\mathcal{Q}(X)) \vee \cdots \vee f^{(-n+1)}(\mathcal{Q}(X))$$

needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$ is equal to the minimal number of open sets of (*) needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$. Also X is contained in the union of all the $I(j_1, j_2, \ldots, j_n)$. Hence

$$M_n(f|X, \mathcal{Q}(X)) \leqslant (K_n(f, \mathcal{C})) \cdot n \cdot k.$$

This implies that

$$\operatorname{ent}(f|X, \mathcal{Q}(X)) \leqslant \lim_{n\to\infty} \frac{1}{n} \left(\ln(K_n(f, \mathcal{C}))\right).$$

Since the diameter of $\mathcal{Q}(X)$ may be taken to be arbitrarily small, we have

$$\operatorname{ent}(f|X) \leqslant \lim_{n\to\infty} \frac{1}{n} \left(\ln(K_n(f, \mathcal{C}))\right).$$

But X contains all nonwandering points of f except for the finite set $\{c_1, c_2, \ldots, c_m\}$. Thus, using Propositions 1–4,

$$\operatorname{ent}(f) = \operatorname{ent}(f|\Omega(f)) = \operatorname{ent}(f|X) \leqslant \lim_{n\to\infty} \frac{1}{n} \left(\ln(K_n(f, \mathcal{C}))\right). \quad \text{Q.E.D.}$$

4. Construction of the sequence f_n. Let f_n be any map in $C^2(S^1, S^1)$ which satisfies properties (1)–(4) of g in §1 and the following:

(5′) There are points $l \in (e, t)$ and $k \in (t, s)$ with $f(l) = f(k) = e$.

(6′) g is one-to-one on each of the intervals (e, l), (l, t), (t, s), and (s, e).

(7′) $f_n(t) \in \operatorname{slsm}(c^{-n})$ where c^{-n} is defined as follows. Let $c^0 = c$. Then for $i = 1, \ldots, n$ let c^{-i} denote the unique inverse image (under f_n) of c^{-i+1} in (e, l). Recall that $\operatorname{slsm}(c^{-n})$ denotes the component of $W^s(c)$ which contains c^{-n}.

The map f_3 is pictured in Figure 2. Let H_1, H_2, H_3, H_4, H_5 be the disjoint

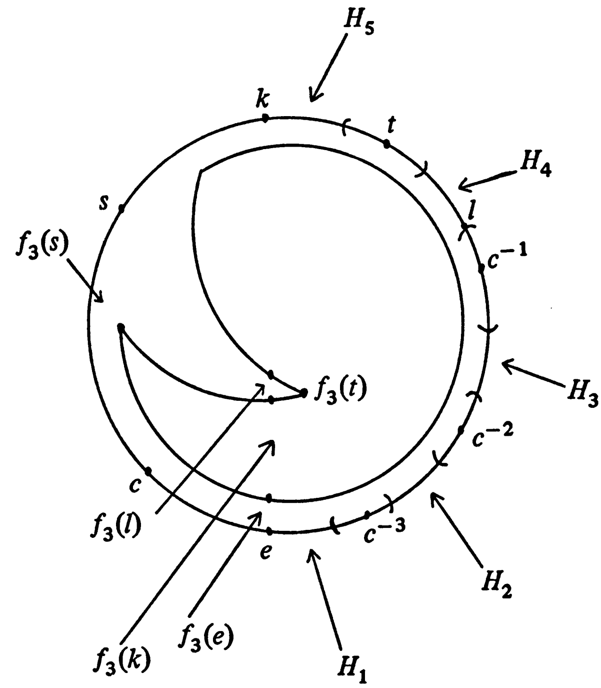

FIGURE 2

closed intervals which form the complement of

$$\operatorname{slsm}(c) \cup \operatorname{slsm}(c^{-1}) \cup \operatorname{slsm}(c^{-2}) \cup \operatorname{slsm}(c^{-3}) \cup \operatorname{slsm}(t).$$

Note that $\operatorname{slsm}(c) = (k, e)$. We can define a 5×5 matrix A_3 by $A_3(i, j) = 1$ if $f_3(H_i) \cap H_j \neq \varnothing$ and $A_3(i, j) = 0$ otherwise. Note that $f_3(H_i) \cap H_j \neq \varnothing$ implies $f_3(H_i) \supset H_j$. It is easy to see from Figure 2 that

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can define a matrix A_n analogously, and A_n is the $(n+2) \times (n+2)$ matrix

$$A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the missing rows have ones on the super diagonal and zeros elsewhere.

The following proposition follows from Theorem D of [2].

PROPOSITION 7. $\text{ent}(f_n) = \ln(\lambda_n)$ *where* λ_n *denotes the largest eigenvalue of* A_n.

THEOREM 8. $\text{ent}(f_n) \to 0$ *as* $n \to \infty$.

PROOF. A straightforward calculation shows that for $n > 3$ the characteristic polynomial of A_n is $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Now λ_n is the largest root of p_n, and it is easy to see that $\lambda_n \to 1$ as $n \to \infty$. The theorem now follows from Proposition 7. Q.E.D.

PROPOSITION 9. *For any neighborhood* N *of* g *in* $C^2(S^1, S^1)$, *there is a map in* N *with positive entropy* (*and hence by Propositions* 1 *and* 2 *infinite nonwandering set*).

PROOF. Let N be any neighborhood of g in $C^2(S^1, S^1)$. There is (for large enough n) a map $h \in N$ satisfying properties (1)–(4) and (5′)–(7′) of the map f_n in the sequence defined above. Hence $\text{ent}(h) = \ln(\lambda_n)$ where λ_n is the largest root of $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Clearly $\lambda_n > 1$, so $ln(\lambda_n) > 0$. Q.E.D.

5. **Proof of Theorem A.**

THEOREM A. *The map* ent: $C^2(S^1, S^1) \to R$ *is continuous at* g.

PROOF. Let $\varepsilon > 0$. Choose N large enough that $\text{ent}(f_N) < \varepsilon$ where f_N is the Nth term of the sequence defined in §4. Choose $\delta > 0$ such that if $d(g, f) < \delta$, where d denotes a metric on $C^2(S^1, S^1)$, then the following hold.

(1) f has an expanding fixed point $e(f)$ and a contracting fixed point $c(f)$ and these are the only fixed points of f.

(2) f preserves orientation at $e(f)$ and $c(f)$.

(3) f has nondegenerate singularities $t(f)$ and $s(f)$ and these are the only singularities of f.

(4) The points $e(f)$, $t(f)$, $s(f)$, $f(s(f))$, and $c(f)$ are distinct and in order on the circle in the counterclockwise direction.

(5) Either (5A) holds or (5B) and (5C) hold.

(5A) $f(t(f)) \in [c(f), e(f)]$ and f is one-to-one on each of the intervals $(e(f), t(f))$, $(t(f), s(f))$ and $(s(f), e(f))$.

(5B) There are points $l(f) \in (e(f), t(f))$ and $k(f) \in (t(f), s(f))$ with $f(l(f)) = f(k(f)) = e(f)$.

Also f is one-to-one on each of the intervals $(e(f), l(f))$, $(l(f), t(f))$, $(t(f), s(f))$, and $(s(f), e(f))$.

(5C) $f(t(f)) \in (e(f), c^{-N}(f))$ where $c^{-N}(f)$ is defined as follows. Let

$c^0(f) = c(f)$. Then for $i = 1, \ldots, N$ let $c^{-i}(f)$ denote the unique inverse image (under f) of $c^{-i+1}(f)$ in $(e(f), l(f))$.

Let $f \in C^2(S^1, S^1)$ with $d(g, f) < \delta$. We will show that $\text{ent}(f) < \varepsilon$. If property (5A) above holds, it follows that $\Omega(f) = \{e(f), c(f)\}$, and $\text{ent}(f) = 0$. Hence we may assume that (5B) and (5C) hold.

We define a collection of proper closed intervals $\mathcal{C}(f) = \{I_1, \ldots, I_{N+2}\}$ as follows. Let $I_1, \ldots, I_N$ be the components of the complement in $[e(f), c^{-1}(f)]$ of

$$\text{slsm}(c^{-1}(f)) \cup \cdots \cup \text{slsm}(c^{-N}(f)).$$

Let $I_{N+1} = [l(f), t(f)]$ and $I_{N+2} = [t(f), k(f)]$. Then if W is the complement in S^1 of

$$\text{slsm}(c(f)) \cup \text{slsm}(c^{-1}(f)) \cup \cdots \cup \text{slsm}(c^{-N}(f))$$

we have $I_1 \cup \cdots \cup I_{N+2} = W$. Hence by Theorem 6,

$$\text{ent}(f) \leqslant \lim_{n\to\infty} \frac{1}{n}\left(\ln\left(K_n(f, \mathcal{C}(f))\right)\right).$$

Let $H_1, \ldots, H_{N+2}$ be the components of the complement in S^1 of the following set (defined with respect to f_N):

$$\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \cdots \cup \text{slsm}(c^{-N}) \cup \text{slsm}(t).$$

Let $h = f_N$ and $\mathfrak{D}(h) = \{H_1, \ldots, H_{N+2}\}$.

It will be helpful for the reader to see Figure 2, in which $N = 3$ and H_1, H_2, H_3, H_4, and H_5 are as indicated. In the case $N = 3$ one may also use Figure 2 for a picture of the intervals I_1, I_2, I_3, I_4, and I_5. To do this, of course, we must replace e, t, c, etc., by $e(f)$, $t(f)$, $c(f)$, etc. Then in the modified figure, I_1, I_2, and I_3 are intervals corresponding to H_1, H_2, and H_3, while $I_4 = [l(f), t(f)]$ and $I_5 = [t(f), k(f)]$.

We may assume that the H_i are numbered as in Figure 2, and the I_i are numbered analogously. We claim that for each positive integer n, $K_n(f, \mathcal{C}(f)) \leqslant K_n(h, \mathfrak{D}(h))$. To prove this claim, suppose that

$$I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cup f^{(-n+1)}(I(j_n)) \neq \varnothing.$$

Then $f(I(j_i)) \cap I(j_{i+1}) \neq \varnothing$ for $i = 1, \ldots, n-1$. By construction, whenever $f(I(i)) \cap I(k) \neq \varnothing$, $A_N(i, k) = 1$ where A_N is the matrix defined in §4. Hence $h(H(j_i)) \supset H(j_{i+1})$ for $i = 1, \ldots, n-1$. This implies that

$$H(j_1) \cap h^{-1}(H(j_2)) \cap \cdots \cap h^{(-n+1)}(H(j_n)) \neq \varnothing.$$

This proves our claim that for each positive integer n, $K_n(f, \mathcal{C}(f)) < K_n(h, \mathfrak{D}(h))$.

Let X be the complement in S^1 of the stable manifold of c (with repsect to $h = f_N$). Let $\mathfrak{D}(X) = \{H_1 \cap X, \ldots, H_{N+2} \cap X\}$. Then $\mathfrak{D}(X)$ is an open cover of X, and for each positive integer n (since the H_i are pairwise disjoint),

$$K_n(h, \mathfrak{D}(h)) = M_n(h|X, \mathfrak{D}(X)).$$

We have for each positive integer n,

$$K_n(f, \mathcal{C}(f)) \leqslant K_n(h, \mathfrak{D}(h)) = M_n(h|X, \mathfrak{D}(X)).$$

Also,

$$\operatorname{ent}(f) \leqslant \lim_{n\to\infty} \frac{1}{n}\left(\ln\left(K_n(f, \mathcal{C}(f))\right)\right).$$

Hence

$$\operatorname{ent}(f) \leqslant \operatorname{ent}(h|X, \mathfrak{D}(X)) \leqslant \operatorname{ent}(h) < \varepsilon. \quad \text{Q.E.D.}$$

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